MATHEMATICS (B. Sc. Part III)

PAPER - V

Uniform Convergence

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1. Point-wise convergence and uniform convergence

Def. Let $\{f_n\}$, $n = 1, 2, 3, \cdots$, be a sequence of real valued functions defined on a non-empty set X i.e. $f_n : X \to R$ for each $n \in N$. To each point $c \in X$ there corresponds a sequence $\{f_n(c)\}$ of real terms $f_1(c), f_2(c), f_3(c) \cdots\cdots$.

We suppose that the sequence $\{f_n(c)\}$ of real terms converges for every $c \in X$.

Let $\{f_n(c)\}$ ~~of real terms~~ converge to $f(c)$.

In this way let the sequences at all points $c, d, e, \cdots$ of X converge to $f(c), f(d), f(e) \cdots$ ——— (1)

Thus we define in a natural way, a real valued function f, with domain X and range the set defined by (1), so that its value $f(d)$ for $d \in X$ is $\lim \{f_n(d)\}$.

Thus $f(x) = \lim \ (f_n(x)), \forall x \in X$ ——— (2)

The function f, thus defined, is known as the limit or the point-wise limit of the sequence (f_n) on X, and the sequence (f_n) is said to be point-wise convergent to f on X.

Def. If the series $\sum f_n$ (of real valued functions f_n defined on X) converges for every point $x \in X$ and we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \forall x \in X \qquad \text{——— (3)}$$

The function f is called the sum or the point-wise sum of the series $\sum f_n$ on X.

Thus if a function f is the point-wise limit of point-wise convergent sequence (f_n) of functions defined on X, then to each $\epsilon > 0$ and to each $x \in X$ there corresponds a positive integer m such that

$$|f_n(x) - f(x)| < \epsilon, \forall n \geq m \qquad \text{——— (4)}$$

of course, if we fix ϵ, the choice of m may depend upon the choice of x

Def. (Uniform convergence): A sequence (f_n) of real valued functions with domain X is said to be uniformly convergent on X to a real valued function f defined on X if for any $\epsilon > 0$ and for all $x \in X$ there corresponds a positive integer m (independent of x but dependent on ϵ) such that for all $x \in X$,

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq m \qquad \text{——— (5)}$$

Also in this situation we say that the function f is the uniform limit of the sequence (f_n).

It is clear that every uniformly convergent sequence is point-wise convergent, and the uniform limit function is the same as the point-wise limit function.

The difference between the point-wise convergence and uniform convergence can be viewed as follows: In case of point-wise convergence, for each $\epsilon > 0$ and for each $x \in X$ there corresponds a positive integer m (depending on ϵ and x both) such that (4) holds for $\forall n \geq m$; whereas in case of uniform convergence, for each $\epsilon > 0$, it is possible to find one positive integer m (dependent on ϵ alone) which will serve for all $x \in X$.

<u>Def.</u> : A series of functions $\sum f_n$ is said to converge uniformly on X if the sequence (s_n) of its partial sums defined by $s_n(x) = \sum_{i=1}^{n} f_i(x)$

converges uniformly on X. Thus a series of functions $\sum f_n$ converges uniformly to f on X if for each $\epsilon > 0$ and for all $x \in X$, there corresponds a positive integer m (independent of x and dependent on ϵ) such that for all x in X,

$$|f_1(x) + f_2(x) + \cdots + f_n(x) - f(x)| < \epsilon \text{ for all } \geq m \quad \text{— (6)}$$

(1) <u>Example of a point-wise convergent sequence which is not uniformly convergent.</u>

consider the sequence (f_n) of real valued functions defined on the real line R by

$$f_n(x) = \frac{nx}{1+n^2x^2} \text{ for all } x \in R$$

for each fixed $x \in R$

$$f_n(x) = \frac{x/n}{1/n^2 + x^2} \to 0 \text{ as } n \to \infty.$$

Thus the sequence (f_n) is point-wise convergent with the function f defined by $f(x) = 0$ for all $x \in R$, as the point-wise limit.

We show that the sequence (f_n) is not uniformly convergent in any interval (a, b) on R with 0 as an interior point.

Suppose that (f_n) is uniformly convergent in (a, b) so that the point-wise limit f is also the uniform limit.

Let $\epsilon > 0$ be given. Then there exists a positive integer m such that $\forall_x \epsilon$ and $\forall n > m$; $\left|\frac{nx}{1+n^2x^2} - 0\right| < \epsilon$

If we take $\epsilon = \frac{1}{4}$ and k an integer $> m$ such that $\frac{1}{k} \in [a, b]$ we find on taking $n = k$ and $x = \frac{1}{k}$ that

$$\frac{nx}{1+n^2x^2} = \frac{k \cdot \frac{1}{k}}{1+k^2 \cdot \frac{1}{k^2}} = \frac{1}{1+1} = \frac{1}{2} \nless \frac{1}{4}$$

Thus we arrive at a contradiction. Therefore, the sequence is not uniformly convergent in the interval $[a, b]$, which contains the point $1/k$.

But since $1/k \to 0$ the interval $[a, b]$ contains 0. Hence the sequence (f_n) is not uniformly convergent in any interval $[a, b]$ containing 0 even though it is point-wise convergent there.

Example 2: Show that the sequence (f_n) where

$$f_n(x) = \frac{1}{x+n}$$ is uniformly convergent in any interval $[0, b]$, $b > 0$.

Solution: For each fixed $x \in [0, b]$,

$$f_n(x) = \frac{1/n}{\frac{x}{n}+1} \to 0 \quad \text{as } n \to \infty$$

Hence the sequence (f_n) converges point-wise to the function f defined by $f(x) = 0$ for all $x \in [0, b]$.

For any $\epsilon > 0$,

$$\left| f_n(x) - f(x) \right| = \left| \frac{1}{x+n} - 0 \right| = \frac{1}{x+n} < \epsilon$$

If $n > (1/\epsilon) - x$. But $(1/\epsilon) - x$ decreases as x increases and its maximum value is $1/\epsilon$ at $x = 0$. Let m be positive integer $\geq 1/\epsilon$, so that for $\epsilon > 0$, there exists m such that

$$|f_n(x) - f(x)| < \epsilon, \ \forall n \geq m.$$

Hence the sequence (f_n) is uniformly convergent in any interval $[0, b]$ with $b > 0$.